Fourier Series and Parseval's Relation

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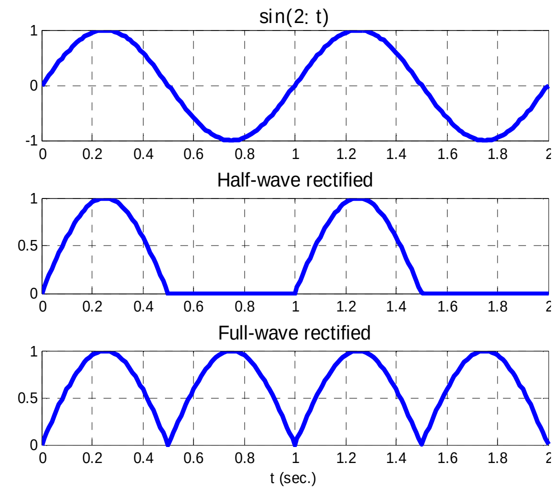
We study the exam problem (EE 301 MT2, Fall2013-14) in some detail to illustrate some connections between Fourier series, Parseval's relation and RMS values.

Q1. (20 pts)

a) The signal $x(t) = \sin(2\pi t)$ is the input to a *half-wave rectifier* circuit with the following input-output relationship:

$$x_{half}(t) = \begin{cases} x(t), & x(t) \geq 0 \\ 0\ , & other \end{cases}.$$

Find the Fourier series coefficients of $x_{half}(t)$. (***Hint***: *You may use Euler's relation to express* $\sin(2\pi t)$.)



b) The signal $x(t) = \sin(2\pi t)$ is the input to a *full-wave rectifier* circuit with the following input-output relationship: $x_{full}(t) = |x(t)| = \begin{cases} x(t), & x(t) \geq 0 \\ -x(t), & other \end{cases}$.

Express the Fourier series coefficients of $x_{full}(t)$ in terms of the Fourier series coefficients found in part (a). (***Note***: *You can call coefficients in part* (a), a_k , *and solve part* (b) *using* a_k *'s* .)

Solution: (The solution is more detailed than it needs to be.)

a) The signal $x(t)$ is periodic with $T = 1$. The Fourier series coefficients can be found through the relation $a_k = \frac{1}{T}\int_0^T x(t)e^{jk\omega_o t}dt$ where $\omega_o = \frac{2\pi}{T} = 2\pi$ rad/sec.

$$
\begin{aligned}
a_k &= \int_0^1 x(t)e^{-j2\pi kt}dt = \int_0^{1/2} \sin(2\pi t)e^{-j2\pi kt}dt \\
&= \frac{1}{2j}\int_0^{1/2}\left(e^{j2\pi t} - e^{-j2\pi t}\right)e^{-j2\pi kt}dt \\
&= \frac{1}{2j}\int_0^{1/2}\left(e^{-j2\pi(k-1)t} - e^{-j2\pi(k+1)t}\right)dt \qquad (*) \\
&= \frac{1}{2j}\left(\frac{e^{-j2\pi(k-1)t}}{-j2\pi(k-1)} - \frac{e^{-j2\pi(k+1)t}}{-j2\pi(k+1)}\right)_{\downarrow t=0}^{\downarrow t=1/2} \qquad (**) \\
&= \frac{1}{4\pi}\left(\frac{e^{-j\pi(k-1)}-1}{k-1} - \frac{e^{-j\pi(k+1)}-1}{k+1}\right)
\end{aligned}
$$

At this point, it should be noted that k index of the sequence a_k is an integer and therefore, $e^{-j\pi k} = (-1)^k$. Substituting $e^{-j\pi k} = (-1)^k$ into the last relation, we can get the following:

$$
\begin{aligned}
a_k &= \frac{1}{4\pi}\left(\frac{e^{-j\pi(k-1)}-1}{k-1} - \frac{e^{-j\pi(k+1)}-1}{k+1}\right) \\
&= \frac{1}{4\pi}\left(-\frac{1+(-1)^k}{k-1} + \frac{1+(-1)^k}{k+1}\right) \\
&= \frac{1+(-1)^k}{4\pi}\left(\frac{1}{k+1} - \frac{1}{k-1}\right) \\
&= \frac{1+(-1)^k}{2\pi}\frac{1}{1-k^2} \\
&= \begin{cases} \dfrac{1}{\pi(1-k^2)} & k: even \\ 0 & k: odd \end{cases}
\end{aligned}
$$

It appears from the last relation that for $k = \pm 1$, we have $a_k = 0$. This is not true, since the equation (**) given above shows that the derived a_k values are only valid when $k \neq \pm 1$. We need to examine the cases of $k = \pm 1$ separately.

From the equation (*), given above, we can express the coefficient a_1 as $a_1 = \frac{1}{2j}\int_0^{1/2}\left(e^{-j2\pi(k-1)t} - e^{-j2\pi(k+1)t}\right)_{\downarrow k=1}dt = \frac{1}{2j}\int_0^{1/2}\left(1 - e^{-j4\pi t}\right)dt = \frac{1}{4\text{j}} = \frac{-j}{4}$. Similarly, we can show that $a_{-1} = \frac{j}{4}$. Given all, the FS coefficients can be written as:

$$
a_k = \begin{cases} \dfrac{1}{\pi(1-k^2)} & k: even \\ -j/4 & k=1 \\ j/4 & k=-1 \\ 0 & k: other \end{cases}.
$$

Now, we can write the Fourier series expansion of the half-wave rectified sinusoidal signal:

$$
\begin{aligned}
x_{Half}(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j\omega_o kt} \\
&= \frac{1}{4j}\left(e^{j\omega_o t} - e^{-j\omega_o t}\right) + \sum_{\substack{k=-\infty,\\ k:even}}^{\infty} a_k e^{j\omega_o kt} \\
&= \frac{1}{4j}\left(2j\sin(\omega_o t)\right) + a_0 + \sum_{\substack{k=2,\\ k:even}}^{\infty} a_k \left(e^{j\omega_o kt} + e^{-j\omega_o kt}\right) \\
&= \frac{\sin(\omega_o t)}{2} + \frac{1}{\pi} + \frac{2}{\pi}\sum_{\substack{k=2,\\ k:even}}^{\infty} \frac{\cos(\omega_o kt)}{1-k^2}
\end{aligned}
$$

It is always useful (and fun) to verify the expansion with a few lines of Matlab code:

```
t=linspace(-2.5,2.5,1024);
FSterms=10;
T=1;
w0=2*pi/T;

out = 1/pi + 1/2*sin(w0*t);  %First few terms

for k=2:2:FSterms, %Remaining terms in the series
    ak = 1/pi/(1-k^2);
    out = out + ak*2*cos(w0*k*t);
end;

plot(t,sin(w0*t),'-.'); hold all
plot(t,out); hold off;
legend('Sine wave','Half wave rectified');
```

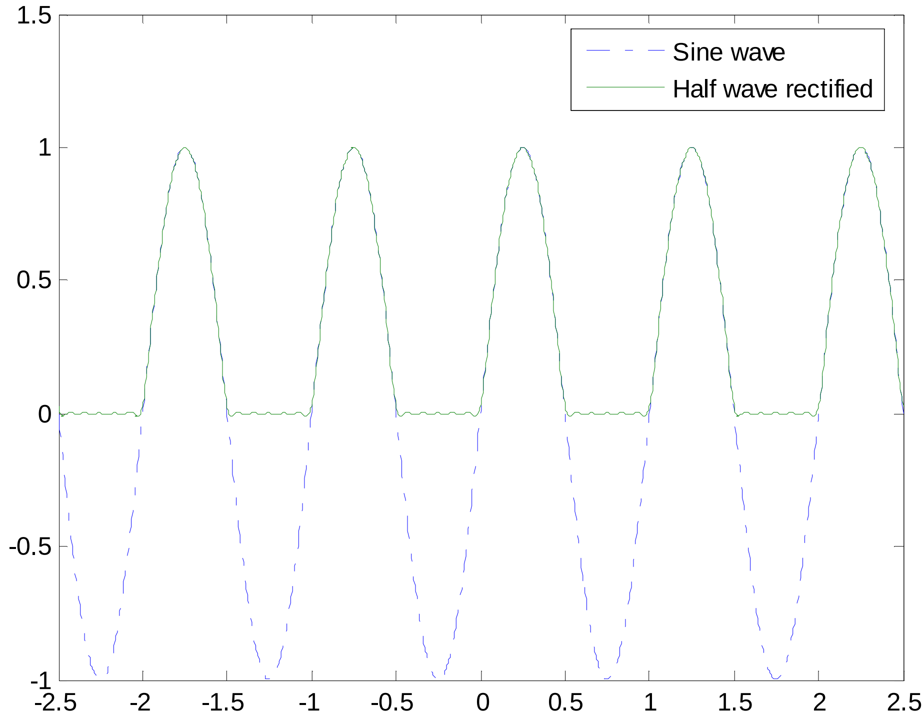


Figure: The plot generated by the given Matlab code

b) If $x_{Half}(t) \leftrightarrow a_k$, the full wave rectified sinusoid signal can be expressed as follows.

$$x_{Full}(t) = x_{Half}(t) + x_{Half}(t-1/2) = \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi t} + \sum_{k=-\infty}^{\infty} a_k e^{jk2\pi(t-1/2)} = \sum_{k=-\infty}^{\infty} a_k (1+e^{-jk2\pi/2}) e^{jk2\pi t}$$

So,

$$x_{Full}(t) = \sum_{k=-\infty}^{\infty} a_k (1+e^{-jk\pi}) e^{jk2\pi t} = \sum_{\substack{k=-\infty \\ k \text{ even}}}^{\infty} 2a_k e^{jk2\pi t}$$

We should be a little careful in this calculation, since the period of $x_{Half}(t)$ is 1 second, while the period of full wave rectified sine signal is 1/2. For $T_0 = 1/2$, the corresponding fundamental frequency is $\omega_0 = \frac{2\pi}{T_0} = 4\pi$.

When writing

$$x_{Full}(t) = \sum_{\substack{k=-\infty \\ k \text{ even}}}^{\infty} 2a_k e^{jk2\pi t}, \text{ with coefficients } c_k = \begin{cases} 2a_k, & \text{for } k \text{ even} \\ 0, & \text{else} \end{cases}, \qquad (1)$$

it is implicitly assumed that $x_{Full}(t)$ is also periodic with 1 second. It is indeed true that $x_{Full}(t)$ is periodic with 1 second, but this is not the fundamental period ($T_0 = 1/2$) and 2π is not the fundamental frequency ($\omega_0 = 4\pi$). 2π is equal to half of the fundamental frequency ; therefore, c_k 's are not the FS coefficients of $x_{Full}(t)$. By substituting $k = 2k'$ in (1), one can easily find the FS coefficients b_k of $x_{Full}(t)$:

$$x_{Full}(t) = \sum_{\substack{k=-\infty \\ k\ even}}^{\infty} 2a_k e^{jk2\pi t} = \sum_{k=-\infty}^{\infty} 2a_{2k} e^{jk4\pi t} = \sum_{k=-\infty}^{\infty} b_k e^{jk4\pi t}, \quad b_k = 2a_{2k}, \ \forall k \qquad (2)$$

[Note that for any periodic signal $x_p(t)$ with fundamental frequency ω_0, one can express $x_p(t)$ as the sum of complex sinusoidals at all frequencies that are multiples of ω_0 / L, for *L*=2, 3, …; however, corresponding coefficients, c_k, are all zero whenever $k \neq mL$ (*m* is an integer). Above example corresponds to *L* = 2.

In order to preserve the uniqueness of the FS representation $x_p(t) \leftrightarrow a_k$, and to safely talk about the *k*'th harmonic power of $x_p(t)$ looking at $|a_k|^2 + |a_{-k}|^2$; only the FS coefficients in the expansion with respect to “complex sinusoidals at all multiples of frequencies ω_0, where ω_0 is the fundamental period” are called the FS coefficients of $x_p(t)$.]

In order to obtain the FS expansion in detail, a_k's found in part (a) can be substituted into (2).

$$\begin{aligned} x_{Full}(t) &= \sum_{k=-\infty}^{\infty} b_k e^{jk4\pi t} \\ &= \sum_{k=-\infty}^{\infty} 2a_{2k} e^{j4\pi kt} \\ &= \frac{2}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{1-4k^2} e^{j4\pi kt} \\ &= \frac{2}{\pi}\left(1 + 2\sum_{k=1}^{\infty} \frac{1}{1-4k^2} \cos(4\pi kt)\right) \end{aligned}$$

It should be noted that the last relation is the conventional expansion of $x_{Full}(t)$ and $\frac{2/\pi}{1-4k^2}$ corresponds to the coefficient of the *k*'th harmonic ($e^{j4\pi kt}$). We can use Matlab to verify our findings.

```
t=linspace(-2.5,2.5,1024);
FSterms=10;

out = 2/pi; %DC term

for k=1:FSterms, %Remaining terms in the series
    ak = 4/pi/(1-4*k^2);
    out = out + ak*cos(4*pi*k*t);
end;

plot(t,sin(w0*t),'-.'); hold all
plot(t,out); hold off;
legend('Sine wave','Full wave rectified');
```

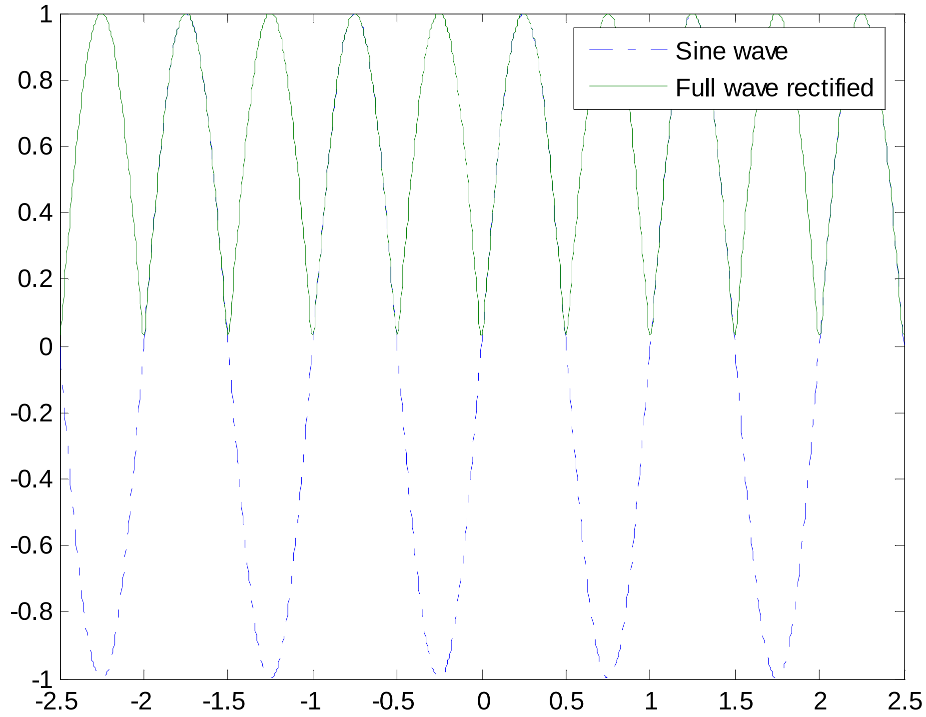


Figure: The plot generated by the given Matlab code

Parseval's Relation, RMS Values and Harmonic Series:

Let's remember the definition for the RMS value of a periodic waveform:

$$x_{RMS} = \sqrt{\frac{1}{T}\int_0^T [x(t)]^2\, dt}\ .$$

In the terminology of electrical engineering, $x(t)$ is considered to be the periodic waveform representing either current or voltage of an $R\,\Omega$ resistor. Then, the average power dissipated over the resistor becomes, $P_{AVG} = \frac{[v_{RMS}]^2}{R}$ or $P_{AVG} = R[i_{RMS}]^2$.

Some of the typical periodic waveforms utilized in circuit applications are given in figure below. (This figure is provided to refresh our memory on RMS calculations.)

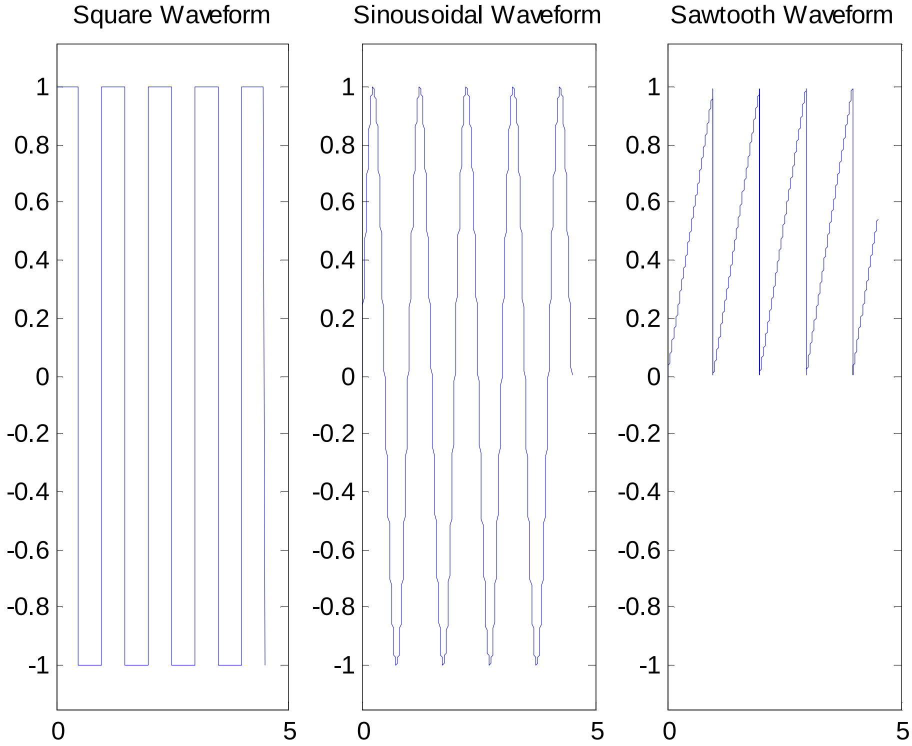


Figure: Some periodic waveforms typically utilized in circuit applications

We know that the square wave with the amplitude A has the RMS value of A; sinusoidal waveform has the RMS value of $\frac{A}{\sqrt{2}}$ and the sawtooth waveform has the value of $\frac{A}{\sqrt{3}}$. The coefficients $\left[1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right]$ scaling the amplitude in this description are called the RMS scaling factors.

In the rest of this section, we calculate the RMS scaling factor for the half wave rectified sinusoidal signal and establish a connection with the Parseval's relation.

We start with the calculation of the RMS value. It should be clear that $x_{Half}(t)$ provides half the average power of a sinusoidal signal; hence its scaling factor should be $\frac{1}{\sqrt{2}} x \frac{1}{\sqrt{2}} = \frac{1}{2}$. We can easily verify this guess with the following calculation: $x_{Half,RMS} = \sqrt{\frac{1}{1}\int_0^{\frac{1}{2}} [\sin(2\pi t)]^2 dt} = \sqrt{\frac{1}{1}\int_0^{\frac{1}{2}} \frac{1-\cos(4\pi t)}{2} dt} = \frac{1}{2}$. (Very similarly, the full wave rectified signal has the RMS scaling factor of $\frac{1}{\sqrt{2}}$.)

Parseval's relation states that $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$. Given our refreshed knowledge on the RMS values, we can also write the Parseval's relation as $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = (x_{RMS})^2$.

Half-wave rectified signal: We have previously found the RMS value of the unit amplitude half-wave rectified signal as $\frac{1}{2}$. In addition, we have also found the FS coefficients of this signal as

$$a_k = \begin{cases} \frac{1}{\pi(1-k^2)} & k: even \\ -j/4 & k=1 \\ j/4 & k=-1 \\ 0 & k: other \end{cases}.$$

Then Parseval's relation $\frac{1}{T}\int_0^T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2 = (x_{RMS})^2$ gives us the following identity:

$$\sum_{k=-\infty}^{\infty} |a_k|^2 = \frac{1}{4}.$$

The summation given above can be rewritten as follows:

$$|a_0|^2 + 2|a_1|^2 + 2\sum_{\substack{k=2,\\ k:even}}^{\infty} |a_k|^2 = \frac{1}{4}.$$

Substituting $|a_0| = \frac{1}{\pi}, |a_1| = \frac{1}{4}$, we reach $\sum_{\substack{k=2,\\ k:even}}^{\infty} |a_k|^2 = \frac{1}{8} - \frac{1}{\pi^2}$. The same relation can also be written as $\frac{2}{\pi^2}\sum_{\substack{k=2,\\ k:even}}^{\infty} \frac{1}{(1-k^2)^2} = \frac{1}{8} - \frac{1}{\pi^2}$ and simplified to the following,

$\sum_{\substack{k=2,\\ k:even}}^{\infty} \frac{1}{(1-k^2)^2} = \frac{\pi^2-8}{16}$ and the summation on the left hand side can be compactly expressed as:

$$\sum_{k=1}^{\infty} \frac{1}{(1-4k^2)^2} = \frac{\pi^2-8}{16}.$$

The end result of this calculation is the summation identity given above. These identities involving reciprocal of integer powers are difficult to prove with elementary means. Therefore, we can not provide any other arguments for the validity of this identity; but we can always use Matlab to numerically examine the correctness of the this identity:

```
>> kvec=1:10;
>> partial_sum = sum(1./(1-4*kvec.^2).^2)

partial_sum =

    0.1168

>> (pi^2-8)/16

ans =

    0.1169
```

It seems that everything is in order.